INTRODUCTION TO GENERALIZED HYPERGEOMETRIC FUNCTION

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Abstract

In mathematics, A Generalized Hypergeometric series is a power series in which the ratio of successive coefficient indexed by n is a rational function of n. If the series is convergent then it is defines a Generalized Hypergeometric Function. Generalized hypergeometric function includes the Gaussian hypergeometric function and confluent hypergeometric function.

Keywords:-

Hypergeometric series, Generalized hypergeometric function, Gaussian and Confluent hypergeometric function, Meijer G-function, Mac Robert E-function.

1. Introduction:-

The term Hypergeometric was first used by Professor John Wallis, in his work Arithmatica Infinitorum (1655) to denote any series which was beyond the ordinary geometric series. In mathematics, a generalized hypergeometric series is a power series in which the ratio of successive coefficients indexed by n is a rational function of n. If the series is convergent then it is defines a generalized hypergeometric function. Generalized hypergeometric function includes the Gaussian hypergeometric function and confluent hypergeometric function as special case. The generalized hypergeometric function is linked to the Meijer G-Function and MacRobert E-Function. The binomial, exponential, logrithm, trignometric and hyperbolic functions are special case of the generalized hypergeometric function.

2. Generalized Hypergeometric Function:-

The theory of generalized hypergeometric function is fundamental in the field of mathematics and mathemtical physics. Most of the functions that occur in analysis are special cases of the hypergeometric functions. Professor John Wallis, in his work, Arithmatica Infinitorum (1655), first used the term hypergeometric to denote any series which was beyond the ordinary geometric series. In fact he studied the series.

$$1 + a + a(a + 1) + a(a + 1)(a + 2) + \ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots (2.1)$$

During the next one hundred and fifty years, many other mathematicians studied the similar series, notably Euler, Vandermonde, Hindenberg etc.

In 1812, C.F. Gauss defined his famous infinite series as follows:

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)\, b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots (2.2)$$

Where $(a)_n$ = a(a+1) .... (a+n–1), $(a)_0$ = 1 is called the Gauss series or the ordinary hypergeometric series. It is denoted by the symbol ${}_2F_1$(a, b, c, z) and named as Gauss hypergeometric function. The series in (2.2) is convergent when $|z| < 1 / G$ also for $z = 1$ provided that Re $(c–a–b) > 0$ and for $z – 1$ if Re $(c – a – b) > –1$.

If in (2.2), we replace z by z/b and let b → ∞, then

$$\frac{(b)_n}{b^n} z^n \to z^n$$

And we arrive at the well-known Kummer's series given below:

$$\sum_{n=0}^{\infty} \frac{(a)_n \; z^n}{(c)_n \; n!} = 1 + \frac{a}{1.c} + \frac{a(a+1)}{1.2.c(c+1)} z^2 + \ldots\ldots$$

.... (2.3)

The series in (2.3) is absolutely convergent for all values of a, c and z (real or complex), excluding c = 0, –1, –2, ..., and is denoted by the symbol ${}_1F_1(a; c; z)$ and named as confluent hypergeometric function.

The Gauss hypergeometric function ${}_2F_1$ and its confluent form ${}_1F_1$ form the case of special functions and have inspired the investigation of a number of other special functions. The work of Barnes, Bailey, Slater, Luke, Erdelyi and many others bear a testimony to the tremendous importance of these functions.

A natural generalization of the ${}_2F_1$ is the generalized hypergeometric function, named as ${}_pF_q$, which is defined as follows:

$${}_pF_q\left[\begin{matrix}\alpha_1, \ldots\ldots, \alpha_p ; z \\ \beta_1, \ldots\ldots, \beta_q\end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!}$$

............(2.4)

The series on the right (2.4) is convergent of all the values of z (real or complex) when $p \leq q$. Also when $p = q+1$, the series is convergent if $| z | < 1$.

It also converges when z = 1, if

$$\mathrm{Re}\left[\sum_{i=1}^{q}(\beta_i) - \sum_{i=1}^{p}(\alpha_i)\right] > 0,$$

And when z = –1, if

$$\mathrm{Re}\left[\sum_{i=1}^{q}(\beta_i) - \sum_{i=1}^{p}(\alpha_i)\right] > -1$$

The binomial, exponential, logarithm, trignometric and hyperbolic functions are special cases of the generalized hypergeometric function. Similarly, orthogonal polynomials such as Legendre functions, Bessel functions, Whittaker function etc. are also its special cases.

T.M. Mac Robert [23] introduced E-function which was studied by many mathematicians in the west particularly. After this in the same time when $p>q+1$, C.S. Meijer in 1936, introduced G-function which revolutionised the world of matheamtics on account of its general nature and wide scope of study. Since then, these functions have attracted attention of a large number of researchers notably Luke (Y.L.), Erdelyi (A.), Al-Salam (W.A.), Bhonsle (B.R.), Mathai (A.M.), Saxena (R.K.), Regab (F.M.), Agarwal (R.P.), Saxena (K.M.), Rathie (C.B.), Buschman (R.G.), Srivastava (H.M.), Sharma (K.C.), Bajpai (S.D.) and several others.

In 1961, Charles Fox [14] introduced a more general function which has since then became well known in the literature as the H-function and covers a large number of functions as its particular cases. This function has been mostly defined and represented by means of the following Mellin-Barnes type of contour integral [16]

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,e_1),\ldots\ldots,(a_p,e_p)\\(b_1,f_1),\ldots\ldots,(b_q,f_q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\frac{\prod_{i=1}^{m}\Gamma(b_i-f_is)\prod_{i=1}^{n}\Gamma(1-a_i+e_is)}{\prod_{i=m+1}^{q}\Gamma(1-b_i+f_is)\prod_{i=n+1}^{p}\Gamma(a_i-e_is)}z^s ds,$$

……………(2.5)

Where m, n, p, q are non-negative integers such that $0\le m\le q$, $0\le n\le p$. The parameters A_1 ………….. A_p; B_1 ………….. B_q are real positive numbers, a_1, …….., a_p; b_1, ……….. b_p, are complex numbers so that the poles of the Gamma functions in the integrand is (5) do not coincide. L is a suitable contour in the s-plane separating the poles of Gamma products with +s and –s in the numerator.

For details on asymptotic expansion and analytic continuation of Fox (1961) H-function, the reader may refer to Braaksma [6].

The importance of the study of the H-function lies in the fact that it includes E, F and G-functions, and all the functions mentioned above as its particular cases.

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